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On a Flat Spread-Sphere Geometry in Odd-dimensional Space.

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§ 1. *Introduction.*

It is a well-known fact that Lie's line-sphere geometry has no analogue in n-dimensional space when n is greater than 3; in fact, in a space $S_n (n > 3)$ there can be no one-to-one correspondence between the ∞^{n+1} spheres and the $\infty^{2(n-1)}$ straight lines; moreover, a contact-transformation which transforms spheres into lines must also transform spheres that touch into lines that intersect. But there is only one necessary and sufficient condition that two spheres shall touch, while the number of conditions that two lines shall intersect is $n-2$. Only if $n = 3$ will therefore a line-sphere geometry be possible.

A generalization of Lie's line-sphere geometry becomes possible if, instead of using the straight line as space-element, we introduce a flat spread (or hyperplane) of $\frac{n-1}{2}$ dimensions. In fact, since to spheres that touch must correspond intersecting flats, the condition for such intersection must be unique. Now if two flats of $\frac{n-1}{2}$ dimensions meet in n-space, the condition for intersection is in general unique; moreover, $\frac{n-1}{2}$ must be an integer. Hence, *a flat spread-sphere geometry is only possible in a space of an odd number of dimensions.* It is also necessary that the number of essential parameters of the flat shall be equal to $n+1$, which is the number of parameters of a sphere.

§ 2. *The Minimal Cone.*

A minimal flat spread of $n-1$ dimensions in S_n (n even) may be defined as one whose direction cosines $A_1, A_2, \ldots, A_n$ satisfy the condition

$$A_1^2 + A_2^2 + \ldots + A_n^2 = 0. \qquad (1)$$

If we put

$$A_1 = \alpha_1 + \beta_1, \quad A_2 = i(\alpha_1 - \beta_1), \quad A_3 = \alpha_2 + \beta_2, \quad \ldots\ldots,$$
$$A_{n-1} = 1 - \sum_1^{\frac{n-2}{2}} \alpha_i\beta_i, \quad A_n = i(1 + \sum_1^{\frac{n-2}{2}} \alpha_i\beta_i), \tag{2}$$

the minimal flat spread, or simply *flat*, may be written

$$(\alpha_1 + \beta_1)X_1 + i(\alpha_1 - \beta_1)X_2 + \ldots\ldots + (1 - \Sigma\,\alpha_i\beta_i)X_{n-1} + i(1 + \Sigma\,\alpha_i\beta_i)X_n + T = 0, \tag{3}$$

where α_i, β_i, T are the parameters of the flat.

If we put $T = F(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}})$ and differentiate (3) with respect to α_i and β_i in succession, we obtain the system

$$\left.\begin{array}{l}
(\alpha_1 + \beta_1)X_1 + i(\alpha_1 - \beta_1)X_2 + \ldots\ldots \\
\qquad + (1 - \Sigma\,\alpha_i\beta_i)X_{n-1} + i(1 + \Sigma\,\alpha_i\beta_i)X_n + F = 0, \\
\qquad X_1 + iX_2 - \beta_1 X_{n-1} + i\beta_1 X_n + F'_{\alpha_1} = 0, \\
\qquad X_1 - iX_2 - \alpha_1 X_{n-1} + i\alpha_1 X_n + F'_{\beta_1} = 0, \\
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\qquad X_{n-3} + iX_{n-2} - \beta_{\frac{n-2}{2}} X_{n-1} + i\beta_{\frac{n-2}{2}} X_n + F'_{\alpha_{\frac{n-2}{2}}} = 0, \\
\qquad X_{n-3} - iX_{n-2} - \alpha_{\frac{n-2}{2}} X_{n-1} + i\alpha_{\frac{n-2}{2}} X_n + F'_{\beta_{\frac{n-2}{2}}} = 0.
\end{array}\right\} \tag{4}$$

Multiplying the second of these equations by α_1, the third by β_1, and so on, we get by adding, and subtracting the sum from the first equation,

$$(1 + \Sigma\,\alpha_i\beta_i)X_{n-1} + i(1 - \Sigma\,\alpha_i\beta_i)X_n + F - \alpha_1 F'_{\alpha_1} - \beta_1 F'_{\beta_1} - \ldots\ldots - F'_{\beta_{\frac{n-2}{2}}},$$

which solved for X_{n-1} gives

$$X_{n-1} = i\frac{(\Sigma\alpha_i\beta_i - 1)}{\Sigma\alpha_i\beta_i + 1}X_n - \frac{F - \Sigma\alpha_i F'_{\alpha_i} - \Sigma\beta_i F'_{\beta_i}}{\Sigma\alpha_i\beta_i + 1}.$$

Substituting this value in (4) we obtain the following system:

$$\left.\begin{array}{l}
= -\dfrac{i(\alpha_1 + \beta_1)}{\Sigma\alpha_i\beta_i + 1}X_n + \dfrac{(\alpha_1 + \beta_1)(\Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i} - F) - (1 + \Sigma\alpha_i\beta_i)(F'_{\alpha_i} + F'_{\beta_i})}{2(\Sigma\alpha_i\beta_i + 1)}, \\
= \dfrac{\alpha_1 - \beta_1}{\Sigma\alpha_i\beta_i + 1}X_n + i\dfrac{(\alpha_1 - \beta_1)(\Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i} - F) + (1 + \Sigma\alpha_i\beta_i)(F'_{\alpha_i} - F'_{\beta_i})}{2(\Sigma\alpha_i\beta_i + 1)}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
{}_2 = \dfrac{\alpha_{\frac{n-2}{2}} - \beta_{\frac{n-2}{2}}}{\Sigma\alpha_i\beta_i + 1}X_n + i\dfrac{(\alpha_{\frac{n-2}{2}} - \beta_{\frac{n-2}{2}})(\Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i} - F) + (1 + \Sigma\alpha_i\beta_i)(F'_{\alpha_{\frac{n-2}{2}}} - F'_{\beta_{\frac{n-2}{2}}})}{2(\Sigma\alpha_i\beta_i + 1)}, \\
{}_1 = \dfrac{i(\Sigma\alpha_i\beta_i - 1)}{\Sigma\alpha_i\beta_i + 1}X_n + \dfrac{\Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i} - F}{\Sigma\alpha_i\beta_i + 1},
\end{array}\right|$$

which is equivalent to (4). If we consider α_i, β_i, F, F'_{α_i}, F'_{β_i} as the coordinates of a minimal line in S_n, we have here the ∞^{2n-3} minimal line of that space. We also observe that through any fixed point there will pass ∞^{n-2} such lines.

If we put

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d, \tag{6}$$

the system (5) reduces to the following:

$$\left.\begin{aligned}
&X_1 + \frac{b_1 + c_1}{2} = \frac{i(\alpha_1 + \beta_1)}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),\\
&X_2 - \frac{i(b_1 - c_1)}{2} = \frac{\alpha_1 - \beta_1}{1 + \Sigma\alpha_i\beta}\left(X_n + \frac{i(a+d)}{2}\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&X_{n-2} - \frac{i(b_{\frac{n-2}{2}} - c_{\frac{n-2}{2}})}{2} = \frac{\alpha_{\frac{n-2}{2}} - \beta_{\frac{n-2}{2}}}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),\\
&X_{n-1} - \frac{a-d}{2} = \frac{i(\Sigma\alpha_i\beta_i - 1)}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),
\end{aligned}\right\} \tag{7}$$

which is a minimal cone having its vertex at the point

$$\left.\begin{aligned}
&\xi_1 = -\frac{b_1 + c_1}{2},\quad \xi_2 = \frac{i(b_1 - c_1)}{2},\quad \dots,\quad \xi_{n-3} = -\frac{b_{\frac{n-2}{2}} + c_{\frac{n-2}{2}}}{2},\\
&\xi_{n-2} = \frac{i(b_{\frac{n-2}{2}} - c_{\frac{n-2}{2}})}{2},\quad \xi_{n-1} = \frac{a-d}{2},\quad \xi_n = -\frac{i(a+d)}{2};
\end{aligned}\right\} \tag{8}$$

in fact, eliminating α_i, β_i from (7), we have

$$(X_1 - \xi_1)^2 + (X_2 - \xi_2)^2 + \dots + (X_n - \xi_n)^2 = 0. \tag{9}$$

We shall now consider α , β_i, F, F'_{α_i}, F'_{β_i} as coordinates of the surface-elements in an $(n-1)$-dimensional space which will be denoted by $\bar{S}_{n-1}$. (Since n is even $\bar{S}_{n-1}$ will be an odd-dimensional space.) It follows then from the above development that *to all the* ∞^n *hyperboloids* (6) *in* $\bar{S}_{n-1}$ *there correspond the* ∞^n *minimal cones in* S_n. *To the* ∞^{n-2} *points of the hyperboloid correspond the* ∞^{n-2} *minimal lines passing through the point* ξ_i; *the addition to* F *in* (5) *of the function* $a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$ *has the effect of translating the minimal hypercomplex* (5) *to the point* ξ_i. *Finally, the system* (5) *establishes a one-to-one correspondence between the* ∞^{2n-3} *minimal lines in* S_n *and the* ∞^{2n-3} *surface-elements* F, α_i, β_i, F_{α_i}, F'_{β_i} *in* $\bar{S}_{n-1}$.*

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*This theorem is a generalization of a corresponding one for $n=4$ in a paper by the author published in *Trans. Amer. Math. Soc.*, Vol. XII, pp. 403–428. The same is true of the theorem on p. 207.

Consider now the isotropic hypercomplex (5). Since we may interpret the quantities $\frac{dX_i}{dX_n}$, $(i = 1, 2, \ldots, n-1)$, as coordinates of the flat space at infinity, we see that the lines of the complex meet the sphere at infinity in a point determined by the values of α_i and β_i. We may therefore consider the quantities

$$\frac{i(\alpha_1 + \beta_1)}{\Sigma\alpha_i\beta_i + 1}, \frac{\alpha_1 - \beta_1}{\Sigma x_i\beta_i + 1}, \ldots, \frac{i(\Sigma\alpha_i\beta_i - 1)}{\Sigma(\alpha_i\beta_i + 1}$$

as coordinates of points on this sphere.

§ 3. *The Focal Surface.*

Let the quantities α_{i0}, β_{i0} determine a generator D of the complex (5). *How many minimal developables of two dimensions can we pass through D?*

To answer this question we proceed as follows: It is evident that we must find a curve determined by the relations

$$\alpha_i = \phi_i(\alpha_{\frac{n-2}{2}}), \quad \beta_k = \psi_k(\alpha_{\frac{n-2}{2}}), \qquad \begin{aligned} i &= 1, 2, \ldots, \frac{n-4}{2}, \\ k &= 1, 2, \ldots, \frac{n-2}{2}, \end{aligned}$$

such that the generators are tangent to it, and moreover such that

$$\alpha_{i0} = \phi_i(\alpha_{\frac{n-2}{2}, 0}), \quad \beta_{k0} = \psi_k(\alpha_{\frac{n-2}{2}, 0}).$$

This curve will then be the edge of regression of the minimal developable M_2. The conditions that two consecutive generators of (5) shall intersect are

$$\left.\begin{aligned} & X_1(d\alpha_1 + d\beta_1) + iX_2(d\alpha_1 - d\beta_1) + \ldots \\ & \qquad - X_{n-1}\Sigma(\alpha_i d\beta_i + \beta_i d\alpha_i) + iX_n\Sigma(\alpha_i d\beta_i + \beta_i d\alpha_i) + dF = 0, \\ & \qquad (-X_{n-1} + iX_n)\, d\beta_1 + dF'_{\alpha_1} = 0, \\ & \qquad (-X_{n-1} + iX_n)\, d\alpha_1 + dF'_{\beta_1} = 0, \\ & \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ & \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ & \qquad (-X_{n-1} + iX_n)\, d\alpha_{\frac{n-2}{2}} + dF'_{\beta_{\frac{n-2}{2}}} = 0. \end{aligned}\right\} \quad (10)$$

Eliminating $X_1, X_2, \ldots, X_n$ from (4) and (10) we have

$$\left.\begin{aligned} & \text{a)} \quad dF - F'_{\alpha_1} d\alpha_1 - F'_{\beta_1} d\beta_1 - \ldots - F'_{\alpha_{\frac{n-2}{2}}} d\alpha_{\frac{n-2}{2}} - F'_{\beta_{\frac{n-2}{2}}} d\beta_{\frac{n-2}{2}} = 0, \\ & \text{b)} \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\beta_i}}{dF'_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\alpha_k}}{dF'_{\beta_{\frac{n-2}{2}}}}, \qquad \begin{aligned} i &= 1, 2, \ldots, \frac{n-4}{2}, \\ k &= 1, 2, \ldots, \frac{n-2}{2}. \end{aligned} \end{aligned}\right\} \quad (11)$$

The first relation is satisfied *à priori*, since we suppose that the function F is finite and continuous, and possesses finite and continuous derivatives in the region α_{i0}, β_{k0}. The remaining $n-3$ relations need special consideration. We shall adopt the following notation, putting

$$a^{(11)}_{ki} = a^{(11)}_{ik} = \frac{\partial^2 F}{\partial\alpha_i\,\partial\alpha_k}, \quad a^{(22)}_{ik} = a^{(22)}_{ki} = \frac{\partial^2 F}{\partial\beta_i\,\partial\beta_k},$$
$$a^{(12)}_{ik} = a^{(21)}_{ki} = \frac{\partial^2 F}{\partial\alpha_i\partial\beta_k}, \quad a^{(21)}_{ik} = a^{(12)}_{ki} = \frac{\partial^2 F}{\partial\beta_i\,\partial\alpha_k}$$

We now write the system (11)b as follows:

$$\left.\begin{aligned}
\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} &= \frac{a^{(11)}_{1k}d\alpha_1 + a^{(11)}_{2k}d\alpha_2 + \ldots + a^{(11)}_{\frac{n-2}{2}}\,d\alpha_{\frac{n-2}{2}} + a^{(21)}_{1k}d\beta_1 + \ldots + a^{(21)}_{\frac{n-2}{2}\,k}\,d\beta_{\frac{n-2}{2}}}{a^{(12)}_{1\frac{n-2}{2}}d\alpha_1 + a^{(12)}_{2\frac{n-2}{2}}\,d\alpha_2 + \ldots + a^{(12)}_{\frac{n-2}{2}\,\frac{n-2}{2}}\,d\alpha_{\frac{n-2}{2}} + a^{(22)}_{1\frac{n-2}{2}}\,d\beta_1 + \ldots + a^{(22)}_{\frac{n-2}{2}\,\frac{n-2}{2}}\,d\beta_{\frac{n-2}{2}}}, \\
\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} &= \frac{a^{(12)}_{1i}d\alpha_1 + a^{(12)}_{2i}d\alpha_2 + \ldots + a^{(12)}_{\frac{n-2}{2}\,i}\,d\alpha_{\frac{n-2}{2}} + a^{(22)}_{11}d\beta_1 + \ldots + a^{(22)}_{\frac{n-2}{2}\,i}\,d\beta_{\frac{n-2}{2}}}{a^{(12)}_{1\frac{n-2}{2}}\,d\alpha_1 + a^{(12)}_{2\frac{n-2}{2}}\,d\alpha_2 + \ldots + a^{(12)}_{\frac{n-2}{2}\,\frac{n-2}{2}}\,d\alpha_{\frac{n-2}{2}} + a^{(22)}_{1\frac{n-2}{2}}\,d\beta_1 + \ldots + a^{(22)}_{\frac{n-2}{2}\,\frac{n-2}{2}}\,d\beta_{\frac{n-2}{2}}}.
\end{aligned}\right\}(12)$$

Introducing a factor of proportionality ρ in this system, we may write it as follows:

$$\left.\begin{aligned}
&(a^{(21)}_{kk} - \rho)\,d\beta_k + a^{(11)}_{1k}d\alpha_1 + \ldots + a^{(11)}_{\frac{n-2}{2}\,k}\,d\alpha_{\frac{n-2}{2}} + a^{(21)}_{1k}d\beta_1 + \ldots \\
&\qquad\qquad + a^{(21)}_{k-1\,k}\,d\beta_{k-1} + a^{(21)}_{k+1\,k}\,d\beta_{k+1} + \ldots = 0, \\
&(a^{(12)}_{ii} - \rho)\,d\alpha_i + a^{(12)}_{1i}d\alpha_1 + \ldots + a^{(12)}_{i-1}\,d\alpha_{i-1} + a^{(12)}_{i+1}\,d\alpha_{i+1} + \ldots \\
&\qquad\qquad + a^{(22)}_{11}d\beta_1 + \ldots = 0, \\
&(a^{(12)}_{1\frac{n-2}{2}}\,d\alpha_1 + a^{(12)}_{2\frac{n-2}{2}}\,d\alpha_2 + \ldots + a^{(22)}_{1\frac{n-2}{2}}\,d\beta_1 + \ldots + (a^{(22)}_{\frac{n-2}{2}\,\frac{n-2}{2}} - \rho)\,d\beta_{\frac{n-2}{2}} = 0.
\end{aligned}\right\}(12')$$

Eliminating $d\alpha_i$ and $d\beta$ from this system we have the following equation of the $(n-2)$-nd degree in ρ:

$$\begin{vmatrix}
a^{(21)}_{11} - \rho & a^{(21)}_{21} \ldots\ldots\ldots\ldots & a^{(11)}_{11} & a^{(11)}_{21} \ldots\ldots\ldots\ldots & a^{(11)}_{\frac{n-2}{2}1} \\
a^{(21)}_{12} & a^{(21)}_{22} - \rho \ldots\ldots\ldots & a^{(11)}_{12} & a^{(11)}_{22} \ldots\ldots\ldots\ldots & a^{(11)}_{\frac{n-2}{2}2} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a^{(21)}_{1\frac{n-2}{2}} & a^{(21)}_{2\frac{n-2}{2}} \ldots\ldots a^{(21)}_{\frac{n-2}{2}\,\frac{n-2}{2}} - \rho & a^{(11)}_{1\frac{n-2}{2}} & a^{(11)}_{2\frac{n-2}{2}} \ldots\ldots & a^{(11)}_{\frac{n-2}{2}\,\frac{n-2}{2}} \\
a^{(22)}_{11} & a^{(22)}_{21} \ldots\ldots\ldots\ldots & a^{(12)}_{11} - \rho & a^{(12)}_{21} \ldots\ldots\ldots & a^{(11)}_{\frac{n-2}{2}1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a^{(22)}_{1\frac{n-2}{2}} & a^{(22)}_{2\frac{n-2}{2}} \ldots\ldots\ldots\ldots & a^{(12)}_{1\frac{n-2}{2}} & a^{(12)}_{2\frac{n-2}{2}} \ldots\ldots\ldots & a^{(12)}_{\frac{n-2}{2}\,\frac{n-2}{2}} - \rho
\end{vmatrix} = 0. \quad (13)$$

If we call the roots $\rho_1, \rho_2, \ldots, \rho_{n-2}$, we see at once that

$$\sum_{1}^{n-2} \rho_i = 2\Sigma \frac{\partial^2 F}{\partial \alpha_i \, \partial \beta_i}. \tag{14}$$

If we now introduce a root of (13) into (12′), we obtain a linear system of $n-3$ differential equations in $n-2$ variables; we have therefore by solving

$$\left.\begin{aligned} \frac{d\beta_k}{\partial \alpha_{\frac{n-2}{2}}} &= f_k(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \\ \frac{d\alpha_i}{\partial \alpha_{\frac{n-2}{2}}} &= \phi_i(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}). \end{aligned}\right\} \tag{15}$$

Since we have assumed F and its derivatives finite and continuous, at least for a certain region in the neighborhood of $\alpha_{i0}, \beta_{k0}, \alpha_{\frac{n-2}{2}0}$, the functions f_k and ϕ_k will in general possess the same property. By Cauchy's existence-theorem [*] there will exist a system of integrals

$$\alpha_i = \phi_i(\alpha_{\frac{n-2}{2}}), \quad \beta_k = \psi_k(\alpha_{\frac{n-2}{2}}) \tag{16}$$

which satisfies (15), and such that for $\alpha_i = a_{i0}$, $\beta_k = \beta_{k0}$ the following relations are satisfied:

$$\alpha_{i0} = \phi_i(\alpha_{\frac{n-2}{2}0}), \quad \beta_{k0} = \psi_k(\alpha_{\frac{n-2}{2}0}). \tag{17}$$

Since ρ has in general $n-2$ distinct values, *there exist $n-2$ minimal developables passing through an arbitrarily chosen generator D of the complex* (5). *These are obtained by solving an equation of the $(n-2)$-nd degree and integrating a system of $n-3$ linear differential equations in $n-2$ variables.* Substituting the values of α_i and β_k from (16) in (5) we obtain the required minimal developables, one for each root of (15). It may happen that for a special value of F equation (13) has multiple roots; in this case the number of developables will be reduced by a number equal to the sum of the multiplicity numbers minus the number of multiple roots.

We may now obtain the focal surface of the complex. From the last of equations (10) we get

$$(-X_{n-1} + iX_n)\, d\alpha_{\frac{n-2}{2}} + \frac{\partial^2 F}{\partial \alpha_1 \partial \beta_{\frac{n-2}{2}}} d\alpha_1 + \ldots + \frac{\partial^2 F}{\partial \alpha_{\frac{n-2}{2}} \partial \beta_{\frac{n-2}{2}}} d\alpha_{\frac{n-2}{2}} + \ldots$$
$$+ \frac{\partial^2 F}{\partial \beta_1 \partial \beta_{\frac{n-2}{2}}} d\beta_1 + \ldots + \frac{\partial^2 F}{\partial^2 \beta_{\frac{n-2}{2}}} d\beta_{\frac{n-2}{2}} = 0,$$

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[*] See Picard, *Traité D'Analyse,* Chapter XI.

which, if we take account of the last of equations (12′), may be written

$$-X_{n-1}+iX_n+\rho_i=0, \tag{18}$$

where ρ_i is any one of the roots of (13). If now we add (18) to the system (4) and solve for $X_1, X_2, \ldots, X_n$, we shall obtain the focal surface which, since ρ has $n-2$ values, possesses in general $n-2$ focal sheets. The equation of the focal surface is:

$$\left.\begin{aligned}
X_1 &= \frac{1}{2}[(\alpha_1+\beta_1)\rho-F'_{\alpha_1}-F'_{\beta_1}],\\
X_2 &= \frac{i}{2}[(\alpha_1-\beta_1)\rho+F'_{\alpha_1}-F'_{\beta_1}],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-1} &= \frac{1}{2}[(1-\Sigma\alpha_i\beta_i)\rho-F+\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}],\\
X_n &= \frac{i}{2}[(1+\Sigma\alpha_i\beta_i)\rho+F-\Sigma\alpha_iF'_{\alpha_i}-\Sigma\beta_iF'_{\beta_i}].
\end{aligned}\right\} \tag{19}$$

We have thus proved the

THEOREM: *The isotropic complex* C_{n-2} *in* S_n *has a focal surface* (19) *which is the envelope of* ∞^{n-3} *minimal developables* M_2 *obtained from the complex by choosing any one of the* ∞^{n-2} *generators and passing through it* $n-2$ *developables. This focal surface has* $n-2$ *sheets. The edges of regression on the surface are minimal curves satisfying the system of differential equations*

$$\left.\begin{aligned}
\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} &= f_k(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \qquad k=1,2,\ldots,\frac{n-2}{2},\\
\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} &= \phi_i(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \qquad i=1,2,\ldots,\frac{n-4}{2}.
\end{aligned}\right\} \tag{20}$$

If we calculate the linear element of the surface (19) we find

$$\overline{d\sigma}_k^2=\sum_1^{n-1}dX_i^2=\sum_1^{\frac{n-2}{2}}(\rho_k\,d\beta-dF'_{\alpha_i})(\rho_k\,d\alpha_i-dF'_{\beta_i}), \tag{21}$$

where the subscript k refers to the focal sheet S_k. If $n=4$, $k=2$ and $d\sigma^2$ is a perfect square, as was proved in a former paper "On Minimal Lines and Congruences in 4-Dimensional Space," *Trans. Amer. Math. Soc.*, Vol. XII, p. 407. The case $n=4$ seems therefore to be exceptional.

§ 4. *The Surface of Reference.*

If we cut the complex C_{n-2} by the space $X_n=0$ we obtain an $(n-2)$-dimensional surface in a new space which we shall denote by S_{n-1}, viz.:

$$\left.\begin{aligned} X_1 &= \frac{(\alpha_1+\beta_1)(\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F)-(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_1}+F'_{\beta_1})}{2(\Sigma\alpha_i\beta_i+1)}, \\ X_2 &= i\,\frac{(\alpha_1-\beta_1)(\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F)+(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_1}-F'_{\beta_1})}{2(\Sigma\alpha_i\beta_i+1)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1} &= \frac{\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F}{\Sigma\alpha_i\beta_i+1}. \end{aligned}\right\} \quad (22)$$

We shall call this surface the *surface of reference* of the complex. It is the envelope of the ∞^{n-2} flats

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots\ldots+(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0, \quad (23)$$

whose direction cosines are

$$\xi_1=\frac{\alpha_1+\beta_1}{1+\Sigma\alpha_i\beta_i}, \quad \xi_1=\frac{i(\alpha_1-\beta_1)}{1+\Sigma\alpha_i\beta_i}, \quad \ldots\ldots, \quad \xi_{n-1}=\frac{1-\Sigma\alpha_i\beta_i}{1+\Sigma\alpha_i\beta_i}; \quad (24)$$

hence, the lines α_i, β_i on the surface of reference correspond to the rectilinear generators of the sphere

$$\sum_1^{n-1}\xi_i^2=1. \quad (25)$$

These generators are minimal straight lines; in fact the linear element of the sphere is found to be

$$d\sigma^2=\frac{2}{(1+\Sigma\alpha_i\beta_i)^2}\left[d\alpha_1\,d\beta_1+d\alpha_2\,d\beta_2+\ldots\ldots+d\alpha_{\frac{n-2}{2}}\,d\beta_{\frac{n-2}{2}}\right].$$

We shall say that the sphere (25) is the *spherical representation* of the surface of reference. Let M be a point on the surface (22) and m its spherical representation, and consider a rectilinear generator D of the sphere; it meets the sphere at infinity in a point P. To it will correspond on the surface a curve which is the locus of points of contact of tangent flats parallel to the generator D; that is to say, the curve of contact of a cone whose vertex is P and which is tangent to the surface; thus, to the rectilinear generator $\alpha_1=\alpha_2=\ldots\ldots=-\alpha_{\frac{n-2}{2}}=\text{const.}$, $\beta_1=\beta_2=\ldots\ldots=\beta_{\frac{n-4}{2}}=\text{const.}$ corresponds on the surface the curve determined by the identical equations. We may also note here that the line-element of the surface of reference takes the form

$$ds^2=\sum_1^{\frac{n-2}{2}} i\,(X_{n-1}\,d\beta_i-dF'_{\alpha_i})\,(X_{n-1}\,d\alpha_i-dF'_{\beta_i}).$$

We may here mention another property of the expression on the right side of (22). If the expression

$$a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$$

be added to F the effect amounts to a translation and a stretching or dilation; in fact we have, on putting for F the function

$$F + a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$$

in (22),

$$\left.\begin{aligned} \bar{X}_1 &= X_1 - \frac{b_1 + c_1}{2} - \frac{a+d}{2}\xi_1, \\ \bar{X}_2 &= X_2 - \frac{i(b_1 - c_1)}{2} - \frac{a+d}{2}\xi_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \bar{X}_{n-1} &= X_{n-1} - \frac{d-a}{2} - \frac{a+d}{2}\xi_{n-1}, \end{aligned}\right\} \quad (26)$$

from which it follows that

$$\left[\bar{X}_1 - \left(X_1 - \frac{b_1 + c_1}{2}\right)\right]^2 + \left[\bar{X}_2 - \left(X_2 - \frac{i(b_1 - c_1)}{2}\right)\right]^2 + \ldots\ldots$$
$$+ \left[\bar{X}_{n-1} - \left(X_{n-1} - \frac{d-a}{2}\right)\right]^2 = \left(\frac{a+d}{2}\right)^2 \Sigma\xi_i^2 = \left(\frac{a+d}{2}\right)^2,$$

and this identity proves the proposition.

We shall now prove the following important

THEOREM: *To the edges of regression of the focal surface* (19) *of the isotropic hypercomplex* C_{n-2} *in* S_n *correspond the lines of curvature on the surface of reference in the space* S_{n-1}.

Proof: The differential equations of the lines of curvature are obtained as follows:* a point $\bar{X}_i$ on the normal at a distance λ from the point of tangency is determined by the following system of equations,

$$\bar{X}_1 = X_1 + \lambda\xi_1, \quad \bar{X}_2 = X_2 + \lambda\xi_2, \quad \ldots\ldots, \quad \bar{X}_{n-1} = X_{n-1} + \lambda\xi_{n-1}, \qquad (27)$$

the ξ's being the direction cosines of the normal. Substituting the values of

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*This method is a generalization of one used by Darboux in his *Leçons*, Vol. I, pp. 238–240, for $n = 3$.

X_i obtained from these equations in (23) and differentiating partially with respect to α_i and β_i, we have the system

$$\left.\begin{array}{l}
(\alpha_1+\beta_1)\overline{X}_1+i\,(\alpha_1-\beta_1)\overline{X}_2+\ldots\ldots+(1-\Sigma\alpha_i\beta_i)\overline{X}_{n-1}+\;F\;=(1+\Sigma\alpha_i\beta_i)\lambda,\\
\quad\overline{X}_1+i\overline{X}_2 \qquad\qquad\qquad -\beta_1\,\overline{X}_{n-1}+\dfrac{\partial F}{\partial\alpha_1}=\beta_1\lambda,\\
\quad\overline{X}_1-i\overline{X}_2 \qquad\qquad\qquad -\alpha_1\,\overline{X}_{n-1}+\dfrac{\partial F}{\partial\beta_1}=\alpha_1\lambda,\\
\qquad\qquad\overline{X}_3+i\overline{X}_4 \qquad -\beta_2\,\overline{X}_{n-1}+\dfrac{\partial F}{\partial\alpha_2}=\beta_2\lambda,\\
\qquad\qquad\overline{X}_3-i\overline{X}_4 \qquad -\alpha_2\,\overline{X}_{n-1}+\dfrac{\partial F}{\partial\beta_2}=\alpha_2\lambda,\\
\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\qquad\qquad\overline{X}_{n-3}+\overline{X}_{n-2}-\beta_{\frac{n-2}{2}}\overline{X}_{n-1}+\dfrac{\partial F}{\partial\alpha_{\frac{n-2}{2}}}=\beta_{\frac{n-2}{2}}\lambda,\\
\qquad\qquad\overline{X}_{n-3}-i\overline{X}_{n-2}-\alpha_{\frac{n-2}{2}}\overline{X}_{n-1}+\dfrac{\partial F}{\partial\beta_{\frac{n-2}{2}}}=\alpha_{\frac{n-2}{2}}\lambda.
\end{array}\right\}\quad(28)$$

If now we wish to obtain the lines of curvature, we must express the condition that there exists a deplacement on the surface such that a point on the normal, for some specific value of λ conveniently chosen, describes a curve tangent to the normal. We must therefore have

$$\frac{d\overline{X}_1}{\alpha_1+\beta_1}=\frac{d\overline{X}_2}{i(\alpha_1-\beta_1}=\ldots\ldots=\frac{d\overline{X}_{n-1}}{1-\Sigma\alpha_i\beta_i}=d\theta. \qquad (29)$$

Differentiating then (28), we get from the first equation

$$\begin{aligned}
&(\alpha_1+\beta_1)d\overline{X}_1+i(\alpha_1-\beta_1)d\overline{X}_2+\ldots\ldots+(1+\Sigma\alpha_i\beta_i)d\overline{X}_{n-1}+dF\\
&\qquad+\overline{X}_1(d\alpha_1+d\beta_1)+i\overline{X}_2(d\alpha_1-d\beta_1)+\ldots\ldots\\
&\qquad\qquad-\overline{X}_{n-1}(\alpha_1\,d\beta_1+\beta_1\,d\alpha_1+\ldots\ldots+\beta_i\,d\alpha_i)\\
&\qquad\qquad\qquad=\lambda d(1+\Sigma\alpha_i\beta_i)+(1+\Sigma\alpha_i\beta_i)\,d\lambda.
\end{aligned}$$

This equation reduces, when account is taken of the remaining equations (28) and the equation (29), to the following simple one:

$$d\theta\,(1+\Sigma\alpha_i\beta_i)=d\lambda. \qquad (30)$$

Differentiating the last $n-3$ equations (28), we have

$$\begin{array}{l}
d\overline{X}_1+id\overline{X}_2-\beta_1\,d\overline{X}_{n-1}-\overline{X}_{n-1}\,d\beta_1+d\,\dfrac{\partial F}{\partial\alpha}=\beta_1\,d\lambda+\lambda d\beta_1,\\
d\overline{X}_1-id\overline{X}_2-\alpha_1\,d\overline{X}_{n-1}-\overline{X}_{n-1}\,d\alpha_1+d\,\dfrac{\partial F}{\partial\beta}=\alpha_1\,d\lambda+\lambda d\alpha_1,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{array}$$

$$d\bar{X}_{n-2} + i d\bar{X}_{n-2} - \beta_{\frac{n-2}{2}} d\bar{X}_{n-1} - \bar{X}_{n-1} d\beta_{\frac{n-2}{2}} + d\frac{\partial F}{\partial \alpha_{\frac{n-2}{2}}} = \beta_{\frac{n-2}{2}} d\lambda + \lambda d\beta_{\frac{n-2}{2}},$$

$$d\bar{X}_{n-2} - i d\bar{X}_{n-2} - \alpha_{\frac{n-2}{2}} d\bar{X}_{n-1} - \bar{X}_{n-1} d\alpha_{\frac{n-2}{2}} + d\frac{\partial F}{\partial \beta_{\frac{n-2}{2}}} = \alpha_{\frac{n-2}{2}} d\lambda + \lambda d\alpha_{\frac{n-2}{2}},$$

or, by virtue of (29) and (30),

$$\begin{aligned} -\bar{X}_{n-1} d\beta_1 + dF'_{\alpha_1} - \lambda d\beta_1 &= 0, \\ -X_{n-1} d\alpha_1 + dF'_{\beta_1} - \lambda d\alpha_1 &= 0, \\ \dots\dots\dots\dots\dots\dots&\dots, \\ \dots\dots\dots\dots\dots\dots&\dots, \\ -\bar{X}_{n-1} d\alpha_{\frac{n-2}{2}} + dF'_{\beta_{\frac{n-2}{2}}} - \lambda d\alpha_{\frac{n-2}{2}} &= 0. \end{aligned}$$

Eliminating λ and X_{n-1} from these equations, we have the following system of differential equations:

$$\frac{d\beta_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\alpha_i}}{dF'_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\beta_i}}{dF'_{\beta_{n-1}}}, \quad \begin{aligned} i &= 1, 2, \dots, \frac{n-4}{2}, \\ k &= 1, 2, \dots, \frac{n-2}{2}. \end{aligned}$$

But this system is precisely the same as the one obtained for the edges of regression on the focal surface in S_n. ((11)b on p. 204.) Hence, the above theorem follows.

Consider now a point M on the focal surface (19) and let E be an edge of regression, L the minimal line tangent to E at M. The line L determines a minimal space Q_{n-1} which is normal to it and at the same time contains L. Cutting this space by the space $X_n = 0$, we obtain a space Q_{n-2} in S_{n-1} which passes through the point P, the intersection of $X_n = 0$ with L. Hence, to the minimal element in S_n corresponds a surface-element in S_{n-1}. To the edge E corresponds a curve E' which is the intersection of the minimal developable determined by E with the space $X_n = 0$. As the line L moves along E the surface-element determined by the point P and the space Q_{n-2} moves along the curve E', which therefore may be considered as an element-band.

We now project M orthogonally on $X_n = 0$ by means of the transformation:[*]

$$x_1 = X_1, \quad x_2 = X_2, \quad \dots, \quad x_{n-1} = X_{n-1}, \quad R = iX_n; \tag{31}$$

the point M is projected into a point m which will describe a curve e. But since MP is perpendicular to Q_{n-1}, mP is also perpendicular to Q_{n-2}. The curve e, the projection of E, is therefore tangent to Pm; that is, e is an evolute of the curve E'. To the focal sheet in S_n will correspond in S_{n-1} a focal sheet of the normal hypercomplex determined by the ∞^{n-2} normals along the lines E' which are lines of curvature on the surface of reference. Since through a point on this surface there pass $n-2$ lines of curvature, the normal complex will have $n-2$ sheets. The principal radii of curvature may be obtained from the system (19) by using the orthogonal transformation (31), viz.:

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[*] For details concerning this transformation see Lie-Scheffers *Berührungstr.*, and also the author's paper in Vol. XII of *Trans. Amer. Math. Soc.*, p. 415, where the case $n = 4$ is treated.

$$\left.\begin{array}{ll} \text{a)} & \left\{\begin{array}{l} x_1 \;\;\, = \frac{1}{2}\left[(\alpha_1+\beta_1)\rho_k - F'_{\alpha_1} - F'_{\beta_1}\right], \\ x_2 \;\;\, = \frac{i}{2}\left[(\alpha_1-\beta_1)\rho_k + F'_{\alpha_1} - F'_{\beta_1}\right], \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ x_{n-1} = \frac{1}{2}\left[(1-\Sigma\alpha_i\beta_i)\rho_k - F + \Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i}\right], \end{array}\right. \\ \text{b)} & R_k \;\;\, = \frac{1}{2}\left[(1+\Sigma\alpha_i\beta_i)\rho_k + F - \Sigma\alpha_i F'_{\alpha_i} - \Sigma\beta_i F'_{\beta_i}\right], \\ & \qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \dots, n-2, \end{array}\right\} \quad (32)$$

which is a generalization of the formulæ given by Darboux in his *Leçons*, Vol. I, p. 245. *To the edges of regression will correspond geodetic lines on the surface* (32)a, *which, in general, has real sheets provided α_i and β_i are chosen conjugate imaginaries.*

§ 5. *A Flat Spread-Sphere Geometry.*

We proved above (p. 203) that to all the ∞^n hyperboloids in space $\bar{S}_{n-1}$

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d \qquad (6)$$

there correspond in S_n the ∞^n minimal cones (7). We shall now employ the generalized Euler's transformation:

$$x_i = F'_{\alpha_i}, \quad y_i = \beta_i, \quad p_i = -\alpha_i, \quad q_i = F'_{\beta_i}, \quad z = F - \Sigma\alpha_i F'_{\alpha_i}, \qquad (33)$$
$$\left(i = 1, 2, \dots, \frac{n-2}{2}\right),$$

by means of which the hyperboloids are transformed into ∞^n flat spreads of $\frac{n-2}{2}$ dimensions

$$x_i = ay_i + b_i, \quad z = \Sigma c\, y_i + d, \quad q_i = -ap_i + c_i; \qquad (34)$$

the last set of equations give the relations between the coordinates of the surface-elements of the flat. We have here a hypercomplex of flats that satisfy the following system of total differential equations:

$$\left.\begin{aligned} & dz - p_1\,dx_1 - p_2\,dx_2 - \dots - q_1\,dy_1 - q_2\,dy_2 - \dots - q_{\frac{n-2}{2}}\,dy_{\frac{n-2}{2}} = 0, \\ & dx_i\,dp_{\frac{n-2}{2}} + dy_i\,dq_{\frac{n-2}{2}} = 0, \qquad i = 1, 2, \dots, \frac{n-2}{2}, \\ & dq_k\,dp_{\frac{n-2}{2}} - dp_k\,dq_{\frac{n-2}{2}} = 0, \qquad k = 1, 2, \dots, \frac{n-4}{2}. \end{aligned}\right\} \quad (35)$$

We may also note that the condition that any two flats shall intersect in a point is unique, viz.:

$$\Sigma(c_i - c_i')\,(b_i - b_i') + (d' - d)\,(a - a') \equiv 0, \qquad (36)$$

and that to a flat (a, c_i, b_i, d) corresponds in S_n a minimal cone whose vertex is

$$\xi_1 = -\frac{b_1 + c_1}{2}, \quad \xi_2 = \frac{i(b_1 - c_1)}{2}, \quad \dots, \quad \xi_{n-1} = \frac{a - d}{2}, \quad \xi_n = -\frac{i(a + d)}{2}. \qquad (37)$$

Further, to flats that intersect correspond two minimal cones that are tangent to each other; that is, they have a minimal line-element in common.

Suppose now we transform the hypercomplex of minimal lines (5) by means of the transformation (33); it takes the following form:

$$\left.\begin{aligned} X_1 &= \frac{-i(y_1 - p_1)}{1 - \Sigma p_i y_i} X_n + \frac{(y_1 - p_1)(\Sigma q_i y_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)}, \\ X_2 &= \frac{-(y_1 + p_1)}{1 - \Sigma p_i y_i} X_n + i\,\frac{(y_1 + p_1)(2 - \Sigma q_i y_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)}, \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ X_{n-1} &= \frac{-i(1 + \Sigma p_i y_i)}{1 - \Sigma p_i y_i} X_n + \frac{\Sigma q_i y_i - z}{1 - \Sigma p_i y_i}, \end{aligned}\right\} \quad (38)$$

and the system (4) becomes

$$\left.\begin{aligned} (1 - \Sigma p_i y_i) X_{n-1} + i(1 + \Sigma p_i y_i) X_n + z - \Sigma q_i y_i &= 0, \\ X_1 + iX_2 - y_1 X_{n-1} + iy_1 X_n + x_1 &= 0, \\ X_1 - iX_2 + p_1 X_{n-1} - ip_1 X_n + q_1 &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots &, \\ \dots\dots\dots\dots\dots\dots\dots\dots &, \\ X_{n-3} - iX_{n-2} + p_{\frac{n-2}{2}} X_{n-1} - ip_{\frac{n-2}{2}} X_n + q_{\frac{n-2}{2}} &= 0. \end{aligned}\right\} \quad (39)$$

The condition that any two consecutive generators of the complex (38) or (39) shall intersect is

$$\left.\begin{array}{ll} \text{a)} & dz - p_1\,dx_1 - p_2\,dx_2 - \ldots\ldots - q_1\,dy_1 - q_2\,dy_2 - \ldots\ldots - q_{\frac{n-2}{2}}\,dy_{\frac{n-2}{2}} = 0, \\ \text{b)} & \begin{cases} dx_i\,dp_{\frac{n-2}{2}} + dy_i\,dq_{\frac{n-2}{2}} = 0, & i = 1, 2, \ldots\ldots, \dfrac{n-2}{2}, \\ dq_k\,dp_{\frac{n-2}{2}} - dp_k\,dq_{\frac{n-2}{2}} = 0, & k = 1, 2, \ldots\ldots, \dfrac{n-4}{2}. \end{cases} \end{array}\right\} \quad (40)$$

Consider now the transformed surface of reference obtained by putting $X_n = 0$ in (38). We have seen that to the edges of regression in S_n correspond lines of curvature on the surface of reference (22) in S_{n-1}. By means of Euler's transformation (33) it is transformed into the following:

$$\left.\begin{array}{ll} X_1 = & \dfrac{(y_1 - p_1)(\Sigma y_i q_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)}, \\ X_2 = & i\,\dfrac{(y_1 + p_1)(z - \Sigma q_i y_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)}, \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1} = & \dfrac{\Sigma y_i q_i - z}{1 - \Sigma p_i y_i}. \end{array}\right\} \quad (41)$$

The lines of curvature on (22) are of course also transformed into a set of curves on (41), which we shall now investigate. By Euler's transformation the system of differential equations ((11)b, p. 204) which defines the system of lines of curvature is transformed into the system (40)b, the integration of which will give us a system of curves on (41) which correspond to the lines of curvature on (22). We shall call these lines *asymptotic curves.** Through any point on the surface pass $n - 2$ such curves corresponding to the $n - 2$ directions, orthogonal to each other on the original surface, which determine the lines of curvature. Their geometric properties are evident from the following considerations:

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*The asymptotic curves defined above are a *special set* belonging to the system

$$\Sigma x_i\,p_i = 0,$$

which has been studied by several others. Cf. Bertini, *Geom. proj. degli Iperspazi*, p. 173. Also, E. O. Lovett, "Contact Transformations between the Fundamental Elements of Space," *Annali di Matematica*, 3d ser., Vol. VII, No. 1, pp. 39–98, and C. L. E. Moore, in a recent paper in the *Annals*, Vol. XIII, p. 94. Of the $\underline{|n-1}$ directions on an $(n-1)$-spread in S_n we have chosen $n-1$ special ones, namely, those that correspond on the transform to the $n-1$ directions of curvature. From the standpoint of a generalized flat-sphere geometry, such as has been worked out in this paper, these lines must be designated as *asymptotic lines.*

The condition that the tangent space to a surface $z = f(x_1, \ldots, x_{\frac{n-2}{2}}, y_1, \ldots, y_{\frac{n-2}{2}})$ shall osculate a curve on the surface is

$$\Sigma(dx_i dp_i + dy_i dq_i) = 0, \tag{42}$$

but from (40)b we have

$$dx_i dp_i + dy_i dq_i = 0, \qquad i = 1, 2, \ldots, \frac{n-2}{2}\,; \tag{43}$$

hence (42) vanishes identically. Therefore, *the hyperplanes of the surface-elements of the surface osculate the asymptotic curves defined by the system of differential equations* (40)b. Again, two consecutive hyperplanes of the surface-elements along the curve contain flat spreads of the form

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d; \tag{44}$$

in fact, we have on substituting in the equation

$$dz = \Sigma p_i dx_i + q_i dy_i,$$

$$\Sigma c_i dy_i \equiv a\Sigma p_i dy_i + \Sigma q_i dy_i,$$

from which it follows that

$$c_i = ap_i + q_i,$$

which system satisfies the differential equations

$$dq_k dp_{\frac{n-2}{2}} - dp_k dq_{\frac{n-2}{2}} = 0. \tag{44$'$}$$

But the systems (45′) and (43) are precisely the same as (40)b. The flats (44) are therefore tangents to the surface along the curve, and any one flat passes through the two consecutive points of tangency determined by the two consecutive tangent hyperplanes which osculate the curve.

Remark. For $n = 4$, the flats (44) become the straight lines $x_1 = ay + b$, $z = cy + d$; the two consecutive tangent planes intersect in a line which passes through the two consecutive points of tangency; the curve becomes therefore an asymptotic line on a surface in ordinary space. The only equation remaining in the system (40)b is $dxdp + dydq = 0$, which is the differential equation of asymptotic lines on a surface.

If we consider the space $S_{n-1}(F_1 \alpha_i \beta_i)$ and a surface $F = \phi(\alpha_i \beta_i)$, the system of curves (11)b, which we shall call Euler's lines,* have by Euler's transformation been transformed into asymptotic lines on the transformed

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*These curves have been considered for 3-space in an article published by the author in *Trans. Amer. Math. Soc.*, Vol. VI, No. 4, pp. 450–471, entitled: "On a Certain System of Conjugate Lines on a Surface Connected with Euler's Transformation."

surface $z = f(x_i, z_i)$. The hyperplanes of the surface-elements of the surface are common with those of the hyperboloids

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d,$$

which are therefore tangents to the surface along the curves. The tangent spaces osculate the curve, since Euler's transformation is a contact-transformation. We shall denote the different spaces used previously as follows:

$$S_n(X_1, X_2, \ldots, X_n), \quad S_{n-1}(X_1, X_2, \ldots, X_{n-1}, P_1, \ldots, P_{n-2}),$$
$$\bar{S}_{n-1}(\alpha_i, \beta, F, F'_{\alpha_i}, F'_{\beta_i}), \quad \bar{\bar{S}}_{n-1}(z, p_i, q_i, x_i, y_i),$$

and we may put in evidence the correspondence between these spaces as follows:

S_n	S_{n-1}	$\bar{S}_{n-1}$	$\bar{\bar{S}}_{n-1}$
Edges of regression on focal surface	Lines of curvature	Euler's lines	Asymptotic lines
∞^n minimal cones	∞^n spheres	∞^n hyperboloids	∞^n flats
$X_n = 0$	Surface of reference	Surface of reference	Surface of reference

We may consider the system (41) as a transformation establishing a one-to-one correspondence between the surface-elements of S_{n-1} and $\bar{\bar{S}}_{n-1}$. It is only necessary to add to it the following:

$$P_1 = -\frac{y_1 - p_1}{1 + \Sigma p_i y_i}, \quad P_2 = \frac{i(y_1 + p_1)}{1 + \Sigma p_i y_i}, \quad \ldots, \quad P_{n-2} = \frac{i(y_{\frac{n-2}{2}} + p_{\frac{n-2}{2}})}{1 + \Sigma p_i y_i}, \quad (41')$$

obtained from (3) by putting $X_n = 0$, by writing it in the form

$$X_{n-1} = -\frac{\alpha_1 + \beta_1}{1 - \Sigma\alpha_i\beta_i} X_1 - \frac{i(\alpha_1 - \beta_1)}{1 - \Sigma\alpha_i\beta_i} X_2 - \ldots - \frac{i(\alpha_{\frac{n-2}{2}} - \beta_{\frac{n-2}{2}})}{1 - \Sigma\alpha_i\beta_i}$$

and using Euler's transformation. From the correspondence between the spaces S_{n-1} and $\bar{\bar{S}}_{n-1}$ it is clear that *the transformation* (41) *and* (41′) *transforms spheres into flats and lines of curvature on a surface in* S_{n-1} *into asymptotic lines on the transformed surface in* $\bar{\bar{S}}_{n-1}$*; it may therefore be regarded as the extension proper of Lie's classical transformation;* in fact, for $n = 4$ we obtain this latter transformation. It may be noted that the rôle which the line-element plays in Lie's line-sphere geometry is here represented by the surface-elements of ∞^n flats; further, although any space S_n has a sphere-geometry, only an odd space can have a sphere-flat geometry of the kind developed in these pages.

We shall study in detail the system (39) when cut by the space $X_n = 0$. We have then, writing the system in a slightly modified form:

$$\left.\begin{array}{ll}\text{a)} & \left\{\begin{array}{l} x_1 = y_1X_{n-1} - (X_1 + iX_2), \\ x_2 = y_2X_{n-1} - (X_3 + iX_4), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ x_{\frac{n-2}{2}} = y_{\frac{n-2}{2}}X_{n-1} - (X_{n-3} + iX_{n-2}), \\ z = -y_1(X_1 - iX_2) - y_2(X_3 - iX_4) - \dots \\ \qquad\qquad - y_{\frac{n-2}{2}}(X_{n-3} - iX_{n-2}) - X_{n-1}; \end{array}\right. \\ \text{b)} & \left\{\begin{array}{l} q_1 = -p_1X_{n-1} - (X_1 - iX_2), \\ q_2 = -p_2X_{n-1} - (X_3 - iX_4), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ q_{\frac{n-2}{2}} = -p_{\frac{n-2}{2}}X_{n-1} - (X_{n-3} - iX_{n-2}). \end{array}\right.\end{array}\right\} \quad (45)$$

If now we consider (45)a, we see that to the ∞^{n-1} points in S_{n-1} there correspond ∞^{n-1} flats of the complex $a + d = 0$; in fact, if we compare (45)a with the system

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d,$$

the statement is at once proven. The point-geometry in S_{n-1} is therefore equivalent to a flat-geometry in $\overline{\overline{S}}_{n-1}$, namely, the geometry of the flat-complex $a + d = 0$. To a point (x_i, y_i, z) in $\overline{\overline{S}}_{n-1}$ corresponds an isotropic flat $I_{\frac{n-2}{2}}$ in in S_{n-1}; in fact, we find from (45)a:

$$\Sigma dX_i^2 = dX_{n-1}[y_1(dX_1 - idX_2) + y_2(dX_3 - idX_4) + \dots + dX_{n-1}] = 0.$$

To a flat of the system

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d$$

corresponds a sphere in S_{n-1} which may be written

$$\left[X_1 + \frac{b_1 + c_1}{2}\right]^2 + \left[X_2 - i\frac{b_1 - c_1}{.2}\right]^2 + \dots + \left[X_{n-1} - \frac{a - d}{2}\right]^2 = \left(\frac{a + d}{2}\right)^2.$$

Denoting the center of this sphere and the radius by $X_1', X_2', \dots, X_n', R$ we have

$$\left.\begin{array}{l} X_1' = -\frac{b_1 + c_1}{2}, \quad X_2' = i\frac{b_1 - c_1}{2}, \quad \dots, \quad X_{n-2} = \frac{i(b_{\frac{n-2}{2}} - c_{\frac{n-2}{2}})}{2}, \\ X_{n-1}' = \frac{a - d}{2}, \quad R = \pm\frac{a + d}{2}, \end{array}\right\} \quad (46)$$

or, solving,

$$\left.\begin{array}{l} b_1 = -(X_1' + iX_2'), \quad b_2 = -(X_3' + iX_3'), \quad \dots, \quad a = X_{n-1}' \pm R; \\ c_1 = -(X_1' - iX_2'), \quad c_2 = -(X_3' - iX_3'), \quad \dots, \quad d = -(X_{n-1}' \mp R), \end{array}\right\} \quad (46')$$

from which it follows that conversely to a given sphere correspond two flats which are conjugates with respect to the flat-complex $a + d = 0$, that is to say, the complex (45)a. The correspondence is therefore a one-to-two.*

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*For a proof of this and similar theorems in the case $n = 4$ (3-space) see Lie, "Über Complexe," *Math. Annalen*, Vol. V, pp. 167–175.

A general complex of flat spreads in $\overline{\overline{S}}_{n-1}$ may be written

$$Aa + \Sigma B_i b_i + \Sigma C_i c_i + Dd + Ee + F = 0, \tag{47}$$

using as coordinates of the complex the following:

$$a,\ b_i,\ c_i,\ d,\ e = ad - \Sigma c_i b_i.$$

We shall call such a complex a *flat-complex*. The ∞^{n-1} flats of (47) are transformed into ∞^{n-1} spheres. We shall prove that *these spheres all intersect a given sphere at a constant angle.**

Proof: Let the sphere be written

$$\Sigma (X_i - X'_i)^2 = R^2. \tag{48}$$

Substituting the values of b_i, c_i, a and d from (46′) in (47) we find that the coordinates of the center and the radius must satisfy the following relation:

$$\Sigma\left(X'_{2i-1} + \frac{B_i + C_i}{2E}\right)^2 + \Sigma\left(X_{2i} + i\frac{B_i - C_i}{2E}\right)^2 + \dots + \left(X'_{n-1} - \frac{A-D}{2E}\right)^2$$
$$= R^2 \pm R\frac{D+A}{E} + \frac{(A-D)^2 + 4\Sigma B_i C_i + 4EF}{4E^2}. \tag{49}$$

Let the coordinates of the fixed sphere be denoted by X'_{10}, X'_{20}, $\dots$, $X'_{n-1\,0}$, R_0. We have then

$$\sum_1^{n-1} (X'_i - X'_{i0})^2 + (iR - iR_0)^2 = 2R_0 R\,(1 + \cos\phi), \tag{50}$$

where ϕ is the angle between the two spheres. This relation is also true for the null-spheres $R = 0$ contained in the complex, and since in this case the null-sphere is a point on the fixed sphere, the equation of this sphere may be obtained by putting $R = 0$ in (50) and considering X'_i as running coordinates. The fixed sphere may therefore be written

$$\Sigma (X'_i - X'_{i0})^2 = R_0^2. \tag{51}$$

But X'_i must satisfy (49) when in this equation R is put equal to zero; hence (51) becomes

$$\sum_1^{\frac{n-2}{2}}\left(X_{2i-1} + \frac{B_i + C_i}{2E}\right)^2 + \sum_1^{n-2}\left[X_{2i} + \frac{i(B_i - C_i)}{2E}\right]^2 + \left(X_{n-1} - \frac{A-D}{2E}\right)^2$$
$$= \frac{(A-D)^2 + 4\Sigma B_i C_i + 4EF}{4E^2}$$

The coordinates of the fixed sphere are therefore

$$X_{2i-1\,0} = -\frac{B_i + C_i}{2E},\ X_{2i0} = -\frac{i(B_i - C_i)}{2E},$$
$$R_0 = \frac{1}{2E}\sqrt{(A-D)^2 + 4\Sigma B_i C_i + 4EF},$$

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*This theorem is a generalization of Lie's theorem on sphere-complexes consisting of all the spheres in 3-space that intersect a given sphere at a constant angle. See Lie-Scheffers *Berührungstr.*, pp. 651–653. For $n = 4$, 3-space, the flat-complex becomes a linear complex, or null-system.

so that the equation (49) of the sphere-complex may be written

$$\Sigma(X'_{2i-1} - X_{2i-1\,0})^2 + \Sigma(X'_{2i} - X_{2i\,0})^2 + (iR - iR_0)^2 - 2RR_0(1 + \cos\phi) = 0,$$

where

$$\cos\phi = \frac{\pm(D + A)}{\sqrt{(A - D)^2 + 4\Sigma B_i C_i + 4EF}};$$

ϕ has therefore the same value for all spheres of the complex. Q. E. D.

If the ∞^{n-1} spheres of the complex all *touch* the given sphere, we have $\phi = 0$; that is, we must have

$$(D + A)^2 = (A - D) + 4\Sigma C_i B_i + 4EF,$$

or,

$$\Sigma B_i C_i + EF - AD = 0. \tag{52}$$

In the space $\bar{\bar{S}}_{n-1}$ we have a flat-complex consisting of all the ∞^{n-1} flats that intersect a given flat. We shall call this complex a *special flat-complex.* We may also extend the idea of *involution* to flat-complexes just as in ordinary space to line-complexes. Let there be given two complexes C and C', and consider the family of complexes $C + kC'_1 = 0$, which we may write

$$(A + kA')a + \Sigma(B_i + kB'_i)b_i + \Sigma(C_i + kC'_i)c_i + (D + kD')d$$
$$+ (E + kE')e + F + kF) = 0.$$

In this family there are two special complexes determined by the quadratic equation

$$k^2L + kM + N = 0, \tag{53}$$

where L, M and N have the following values:

$$L = \Sigma C'_i B'_i + E'F' - A'D',$$
$$M = \Sigma(C_i B'_i + C'_i B_i) + E'F + EF' - A'D - AD',$$
$$N = \Sigma C_i B_i + EF - AD.$$

If now k_1 and k_2 be the roots of the quadratic equation (53), we shall say that the complexes are in involution whenever the double ratio $(0, k_1, \infty, k_2)$ equals —1, that is, whenever $k_1 + k_2 = 0$; *hence the condition for involution is*

$$\Sigma(C_i B'_i + C'_i B_i) + E'F + EF' - A'D - AD' = 0.$$

If $\phi = 90°$, we have $A + D = 0$, and the complexes

$$Aa + \Sigma B_i b + \Sigma C_i c_i - Ad + Ee + F = 0$$

and $a + d = 0$ are in involution.

Conversely, the ∞^{n-1} spheres which cut a given sphere at a constant angle are transformed into the flats of two complexes which are conjugate to each other with respect to the complex $a + d = 0$. The proof is not difficult and need not be given here.

The flats of the linear complex $a + d = 0$ which intersect any given flat f_1 (and also its reciprocal, or polar f_2 with respect to the complex) are transformed into the points of a sphere; the points of the flats f_1 and f_2 are transformed into the flat spread generators of the sphere. These are minimal flats of $\frac{n-2}{2}$ dimensions, and they are seen to be divided into two distinct families just like the rectilinear generators of the sphere. Any two generators of different families intersect in a point on the sphere which is the image of the flat joining two points on f_1 and f_2.

A surface-element in S_{n-1} contains ∞^1 flats passing through its point; to these flats are associated another set of ∞^1 flats which are reciprocal with respect to the complex $a + d = 0$, and which also pass through the point of a second surface-element. In S_{n-1} we get two intersecting minimal flats and ∞^1 spheres touching each other at this point. Hence, to a surface F and all the ∞^1 flats which touch it at a given point there correspond in S_{n-1} a surface and all the ∞^1 spheres which touch it at a point.

If a flat has a singly infinite number of points in common with the surface, we get in S_{n-1} a surface which is touched by a sphere along a curve. To a surface generated by moving a flat so that it is tangent to a given curve (corresponding to developables in ordinary space) corresponds the envelope of ∞^1 spheres which move in such a way that two consecutive spheres touch. To "ruled" surfaces correspond canal surfaces in S_{n-1}, etc. It is evident from the above that the geometry of flat-complexes in $\bar{\bar{S}}_{n-1}$ is equivalent to the geometry of sphere-complexes in S_{n-1} ($n-1$ odd).

We found above that there are two special flat-complexes contained in a family $C + kC' = 0$. Each of these has a flat as directrix. If we consider all the flats which intersect these two directrices, we get a complex of ∞^{n-2} flats which we may denote by the symbol C_{n-2}. Hence *to the ∞^{n-2} flats which intersect two fixed flats there correspond the ∞^{n-2} spheres which intersect two fixed spheres at constant angles ϕ_1 and ϕ_2.*

We may continue in this manner building up a theory of sphere- and flat-complexes, just as has been done by Lie in his paper on Complexes in *Mathematische Annalen*, Vol. V, for ordinary space.

PART II.

§ 1. *The Flat-Sphere Transformations.*

It remains now to find all the contact-transformations that have the same property as (41) and (41′). We know that the transformations of the conform group in S_n leave invariant the ∞^n minimal cones and that to this group there corresponds in S_{n-1} a group of contact-transformations which leaves invariant the ∞^n spheres of S_{n-1}. We may obtain the characteristic functions of this group in the following manner:

The conform group in S_n is*

$$\frac{\partial f}{\partial X_\nu},\quad X_\mu\frac{\partial f}{\partial X_\nu}-X_\nu\frac{\partial f}{\partial X_\mu},\quad \sum_1^n X_j\frac{\partial f}{\partial X_j},\quad 2X_\nu\sum_1^n X_j\frac{\partial f}{\partial X_\nu}-\frac{\partial f}{\partial X}\sum_1^n X_j^2, \tag{1}$$

$$\nu,\ \mu=1,\ 2,\ \ldots\ldots,\ n.$$

The sphere corresponding to a minimal cone whose vertex is $(X_1, X_2, \ldots\ldots, X_n)$ we found to be (p. 203)

$$(x_1-X_1)^2+(x_2-X_2)^2+\ldots\ldots+(x_{n-1}-X_{n-1})^2+X_n^2=0, \tag{2}$$

which it will be convenient to put in the following form, introducing auxiliary variables $\phi_1, \phi_2, \ldots\ldots, \phi_{n-1}$,

$$x_1=X_1+iX_n\cos\phi_1,\quad x_2=X_2+iX_n\cos\phi_2,\quad\ldots\ldots,\quad x_{n-1}=X_{n-1}+iX_n\cos\phi_{n-1}, \tag{3}$$

where $\sum\limits_1^{n-1}\cos^2\phi_i=1$.

Differentiating (2) partially with respect to $x_1, x_2, \ldots\ldots, x_{n-2}$ in succession, we have

$$\left.\begin{array}{r} x_1-X_1+(x_{n-1}-X_{n-1})p_1=0,\quad x_2-X_2+(x_{n-1}-X_{n-1})p_2=0,\quad\ldots\ldots,\\ x_{n-2}-X_{n-2}+(x_{n-1}-X_{n-1})p_{n-2}=0.\end{array}\right\} \tag{4}$$

From (3) and (4) we find

$$p_1=-\frac{\cos\phi_1}{\cos\phi_{n-1}},\quad p_2=-\frac{\cos\phi_2}{\cos\phi_{n-1}},\quad\ldots\ldots,\quad p_{n-2}=-\frac{\cos\phi_{n-2}}{\cos\phi_{n-1}}, \tag{5}$$

from which we obtain the following equations:

$$\left.\begin{array}{c} \cos\phi_1=\dfrac{-p_1}{\sqrt{1+\Sigma p_i^2}},\quad \cos\phi_2=\dfrac{-p_2}{\sqrt{1+\Sigma p_i^2}},\quad\ldots\ldots,\quad \cos\phi_{n-2}=\dfrac{-p_{n-2}}{\sqrt{1+\Sigma p_i^2}},\\ 1+\Sigma p_i^2=\dfrac{1}{\cos^2\phi_{n-1}}.\end{array}\right\} \tag{5'}$$

From (3) and (5) we have, making x_i, X_i and ϕ_i vary,

$$\left.\begin{array}{l} \delta x_1\ \ =\delta X_1+i\cos\phi_1\,\delta X_n-iX_n\sin\phi_1\,\delta\phi_1,\\ \delta x_2\ \ =\delta X_2+i\cos\phi_2\,\delta X_n-iX_n\sin\phi_2\,\delta\phi_2,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \delta x_{n-1}=\delta X_{n-1}+i\cos\phi_{n-1}\,\delta X_n-iX_n\sin\phi_{n-1}\,\delta\phi_{n-1},\end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l} \delta p_1\ \ =\sqrt{1+\Sigma p_i^2}\sin\phi_1\,\delta\phi_1-\dfrac{\cos\phi_1\sum\limits_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}},\\ \delta p_2\ \ =\sqrt{1+\Sigma p_i^2}\sin\phi_2\,\delta\phi_2-\dfrac{\cos\phi_2\sum\limits_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \delta p_{n-2}=\sqrt{1+\Sigma p_i^2}\sin\phi_{n-2}\,\delta\phi_{n-2}-\dfrac{\cos\phi_{n-2}\sum\limits_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}}.\end{array}\right\} \tag{7}$$

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* See Lie, *Theorie der Transformationsgruppen,* Vol. III, p. 334.

If now by means of the equations (7) we eliminate $\delta\phi_i$ from (6) we have the following system of equations:

$$\left.\begin{aligned}
\delta x_1 &= \delta X_1 + i\cos\phi_1\,\delta X_n - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\,\Sigma p_i\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= \delta X_2 + i\cos\phi_2\,\delta X_n - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\delta x_{n-1} &= dX_{n-1} + i\cos\phi_{n-1}\,\delta X_n - i\,X_n\frac{\overset{n-2}{\Sigma}\,p_i\delta p_i}{[1+\Sigma p_i^2]^{\frac{3}{2}}}\,.
\end{aligned}\right\}\quad(8)$$

Introducing in (8) the values of $\delta X_1, \delta X_2, \ldots, \delta X_n$ obtained from (1) and eliminating $X_1, X_2, \ldots, X_{n-1}$ from the resulting equations, we obtain a set of equations which must be identically satisfied for all values of X_n. We thus obtain the values of the increments $\delta x_1, \ldots, \delta x_{n-1}, \delta p_1, \ldots, \delta p_{n-2}$. Thus, suppose we take the transformation

$$Uf = X_2\frac{\partial f}{\partial X_1} - X_1\frac{\partial f}{\partial X_2}\,;$$

we have here

$$\delta X_1 = X_2\,\delta t, \quad \delta X_2 = -X_1\,\delta t, \quad \delta X_i = 0, \quad (i = 3, 4, \ldots, n).$$

The system (8) now becomes

$$\begin{aligned}
\delta x_1 &= \quad X_2\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\Sigma p_i\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= -X_1\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\Sigma p_i\delta p_i}{1+\Sigma p_i^2},\\
\delta x_3 &= -iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_3 + \cos\phi_3\Sigma p_i\delta p_i}{1+\Sigma p_i^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\delta x_{n-1} &= -iX_n\frac{\Sigma p_i\delta p_i}{[1+\Sigma p_i^2]^{\frac{3}{2}}}.
\end{aligned}$$

Introducing in this system the values of X_1 and X_2 from (3), we have

$$\begin{aligned}
\delta x_1 &= \quad (x_2 - iX_n\cos\phi_2)\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\Sigma p_i\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= -(x_1 - iX_n\cos\phi_1)\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\delta x_{n-1} &= -iX_n\,\frac{\Sigma p_i\delta p_i}{(1+\Sigma p_i^2)^{\frac{3}{2}}}.
\end{aligned}$$

From the last equation we get $\Sigma p_i \delta p_i = 0$, and from the first $n-2$

$$\begin{array}{llll} \delta x_1 = x_2\,\delta t, & \delta x_2 = -x_1\,\delta t, & \delta x_i = 0, & i = 3, \ldots, n-1, \\ \delta p_1 = p_2\,\delta t, & \delta p_2 = -p_1\,\delta t, & \delta p_i = 0, & i = 3, \ldots, n-1, \end{array}$$

and the characteristic function

$$W = p_1 \xi_1 + p_2 \xi_2 + \ldots + p_{n-2}\xi_{n-2} - \xi_{n-1} = p_1 x_2 - p_2 x_1.$$

In this manner we obtain the following characteristic functions of the required group of contact-transformations, using capital letters in order to conform with the notation of the formulæ (41),

$$\left.\begin{array}{l} 1, \quad P_1, P_2, \ldots, P_{n-2}, \quad X_\mu P_\nu - X_\nu P_\mu, \quad X_\nu + P_\nu X_{n-1}, \\ X_\nu\sqrt{1+\Sigma P_j^2}, \quad X_{n-1}\sqrt{1+\Sigma P_j^2}, \quad \sqrt{1+\Sigma P_j^2}, \quad X_{n-1} - \sum\limits_1^{n-2} P_j X_j, \\ 2X_n X_{n-1} + P_\nu \sum\limits_1^{n-1} X_j^2 - 2X_\nu \sum\limits_1^{n-2} P_j X_j, \\ -2X_{n-1}\Sigma P_j X_j - \sum\limits_1^{n-2} X_j^2 + X_{n-1}^2, \quad \sum\limits_1^{n-1} X_j^2 \sqrt{1+\Sigma P_j^2}. \end{array}\right\} \quad (9)$$

There exist therefore, in the space S_{n-1}, $\frac{n+1\cdot n+2}{2}$ *infinitesimal contact-transformations which leave invariant the* ∞^n *spheres of that space.* This theorem has been proved by Lie.[*]

If now we superpose the generalized Lie's transformation (41) and (41′) on each one of these transformations, we shall obtain a group of transformations in $\overline{\overline{S}}_{n-1}$ which leaves invariant the ∞^n flats (44) and transforms the asymptotic lines of an $(n-2)$-spread into asymptotic lines on the transformed spread. We shall show how to obtain the characteristic functions of this group, which is similar to the group whose characteristic functions are given by (9).

If $U(X_i, P_i)$ represents any characteristic function of the set (9) and $V(x_i, y_i, p_i, q_i, z)$ the required function, the following relation exists between them :[†]

$$\rho V(x_i, y_i, p_i, q_i, z) = U(X_i, P_i),$$

where ρ is determined by the relation

$$dX_{n-1} - \sum_1^{n-2} P_j\,dX_j = \rho\,(dz - \Sigma\, p_j\,dx_j - \Sigma\, q_j\,dy_j).$$

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[*] Lie, *Götting. Nachr.*, May, 1871, ***Math. Ann.***, Bd. V (1872). The above method of deriving the characteristic functions of the general group of contact-transformations which change spheres into spheres is a generalization of the one used in Lie-Scheffers *Berührungstr.*, pp. 441–443, for $n=3$, or, rather, a generalization of the method when reversed, that is, going from 3-space back to 2-space.

[†] Lie, *Theorie der Transformationsgruppen*, Vol. II, pp. 527–528.

From the equations (41) and (41′) we find, after an easy calculation, that $\rho = \frac{1}{1 + \Sigma p_j y_j}$, so that we have

$$V(x_i, y_i, p_i, q_i, z) = U(X_i, P_i)\,(1 + \Sigma p_j y_j),$$

and V may be obtained by expressing U in terms of the coordinates x_i, y_i, p_i, q_i and z. We shall not repeat the rather long and tedious calculations here, but write the functions in order corresponding to (9):

$$\left.\begin{array}{c}
1, \quad y_i - p_i, \quad y_i + p_i, \quad x_k p_k + y_k q_k, \\
x_k y_i - x_i y_k + q_i p_k - p_i q_k + p_i x_k - p_k x_i + q_i y_k - q_k y_i, \\
x_i y_k - y_i x_k + p_i q_k - q_i p_k + p_i x_k - p_k x_i + q_i y_k - y_i q_k, \\
y_i x_k - x_i y_k + p_i q_k - q_i p_k - p_i x_k - q_k y_i - q_i y_k - x_i p_k, \\
x_i y_k - x_k y_i + q_i p_k - q_k p_i - p_i x_k - q_i y_k - q_k y_i - x_i p_k, \\
z y_k - y_k \Sigma q_j y_j - x_k \Sigma p_j y_j - x_k - z p_k + p_k \Sigma q_j y_j - p_k \Sigma p_j y_j - p_k, \\
-z y_k + y_k \Sigma q_j y_j + x_k \Sigma p_j y_j + x_k - z p_k + p_k \Sigma q_j y_j - p_k \Sigma p_j y_j - p_k, \\
-z y_k + y_k \Sigma y_j q_j + x_k \Sigma p_j y_j - x_k + z p_k - p_k \Sigma y_j q_j + q_k \Sigma p_j y_j - q_k, \\
-z y_k + y_k \Sigma y_j q_j + x_k \Sigma p_j y_j - x_k - z p_k + p_k \Sigma y_j q_j - q_k \Sigma p_j y_j + q_k, \\
\Sigma q_j y_j - z, \quad 1 - \Sigma p_j y_j, \quad \Sigma x_j p_j - z, \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j + z q_k + p_k \Sigma x_j p_j - q_k \Sigma x_j p_j, \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j - z q_k - p_k \Sigma x_j q_j + q_k \Sigma x_j p_j, \\
z^2 - z\Sigma q_j y_j - z\Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j - \Sigma x_j q_j, \\
z^2 - z\Sigma q_j y_j - z\Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j + \Sigma x_j q_j, \\
\left(i, k = 1, 2, \ldots, \frac{n-2}{2}, i \neq k\right).
\end{array}\right\} \quad (10)$$

By properly combining, that is, adding and subtracting, we may reduce this set to the following simpler one:

$$\left.\begin{array}{l}
1, \quad p_i, \quad y_i, \quad x_k p_k + y_k q_k, \quad x_k y_i - x_i y_k, \quad q_i p_k - p_i q_k, \\
p_i x_k + q_i y_k, \quad x_i p_k + y_i q_k, \quad x_k, \quad q_k, \quad z y_k - y_k \Sigma q_j y_j - x_k \Sigma p_j y_j, \\
z p_k - p_k \Sigma q_j y_j + q_k \Sigma p_j y_j, \quad \Sigma q_j y_j, \quad \Sigma p_j y_j, \quad 2z - \Sigma(x_j p_j + q_j y_j), \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j, \quad z q_k + p_k \Sigma x_j q_j - q_k \Sigma x_j p_j, \\
z^2 - z\Sigma q_j y_j - z\Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j \Sigma x_j q_j.
\end{array}\right\} \quad (11)$$

As a check on the accuracy of these calculations we may observe that for $n = 4$ we get the following functions:

$$\begin{array}{c}
1, \quad p, \quad y, \quad xp + qy, \quad x, \quad q, \quad zy - y^2 q - xyp, \quad zp, \quad qy, \quad py, \\
2z - xp - qy, \quad zx - xyq - x^2 p, \quad zq, \quad z^2 - zyq - zxp, \quad xq,
\end{array}$$

which are the characteristic functions of the projective group in 3-space considered as a contact-transformation.

The corresponding group of contact-transformations is now

$$
\left.\begin{aligned}
&\frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial x_i}, \quad \frac{\partial f}{\partial q_i} + y_i \frac{\partial f}{\partial z}, \quad x_k \frac{\partial f}{\partial x_k} + y_k \frac{\partial f}{\partial y_k} - p_k \frac{\partial f}{\partial p_k} - q_k \frac{\partial f}{\partial q_k}, \\
&x_k \frac{\partial f}{\partial q_i} - x_i \frac{\partial f}{\partial q_k} + y_i \frac{\partial f}{\partial p_k} - y_k \frac{\partial f}{\partial p_i} + (x_k y_i - x_i y_k) \frac{\partial f}{\partial z}, \\
&q_i \frac{\partial f}{\partial x_k} - q_k \frac{\partial f}{\partial x_i} + p_k \frac{\partial f}{\partial y_i} - p_i \frac{\partial f}{\partial y_k} + (q_i p_k - q_k p_i) \frac{\partial f}{\partial z}, \\
&x_k \frac{\partial f}{\partial x_i} + y_k \frac{\partial f}{\partial y_i} - p_i \frac{\partial f}{\partial p_k} - q_i \frac{\partial f}{\partial q_k}, \quad x_i \frac{\partial f}{\partial x_k} + y_i \frac{\partial f}{\partial y_k} - p_k \frac{\partial f}{\partial p_i} \\
&- q_k \frac{\partial f}{\partial q_i}, \quad \frac{\partial f}{\partial p_i} + x_i \frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial y_i}, \quad x_k \Sigma y_j \frac{\partial f}{\partial x_j} + y_k \Sigma y_j \frac{\partial f}{\partial y_j} \\
&+ y_k z \frac{\partial f}{\partial z} - y_k \Sigma p_j \frac{\partial f}{\partial p_j} + x_k \Sigma p_j \frac{\partial f}{\partial q_j} + (\Sigma q_j y_j - q_k y_k - z) \frac{\partial f}{\partial q_k} \\
&+ \Sigma p_j y_j \frac{\partial f}{\partial p_k}, \quad q_k \Sigma y_j \frac{\partial f}{\partial x_j} - p_k \Sigma y_j \frac{\partial f}{\partial y_j} + (z - \Sigma q_j y_j) \frac{\partial f}{\partial x_k} \\
&+ \Sigma p_j y_j \frac{\partial f}{\partial y_k} + (q_k \Sigma y_j p_j - p_k \Sigma y_j q_j) \frac{\partial f}{\partial z} - p_k \Sigma p_j \frac{\partial f}{\partial p_j} - q_k \Sigma p_j \frac{\partial f}{\partial q_j}, \\
&\Sigma y_j \frac{\partial f}{\partial y_j} - \Sigma q_j \frac{\partial f}{\partial q_j}, \quad \Sigma y_j \frac{\partial f}{\partial x_j} - \Sigma p_j \frac{\partial f}{\partial q_j}, \quad \Sigma \Big(x_j \frac{\partial f}{\partial x_j} + y_j \frac{\partial f}{\partial y_j} \\
&+ p_j \frac{\partial f}{\partial p_j} + q_j \frac{\partial f}{\partial q_j} \Big) + 2z \frac{\partial f}{\partial z}, \quad p_k \Sigma x_j \frac{\partial f}{\partial y_j} - q_k \Sigma x_j \frac{\partial f}{\partial x_j} \\
&+ (z - \Sigma x_j p_j) \frac{\partial f}{\partial y_k} + \Sigma q_j x_j \frac{\partial f}{\partial x_k} + (p_k \Sigma x_j q_j - q_k \Sigma x_j p_j) \frac{\partial f}{\partial z} \\
&- q_k \Sigma q_j \frac{\partial f}{\partial q_j} - p_k \Sigma q_j \frac{\partial f}{\partial p_j}, \quad y_k \Sigma x_j \frac{\partial f}{\partial y_j} + x_k \Sigma x_j \frac{\partial f}{\partial x_j} + x_k z \frac{\partial f}{\partial z} \\
&- x_k \Sigma q_j \frac{\partial f}{\partial q_j} + y_k \Sigma q_j \frac{\partial f}{\partial q_j} + (\Sigma p_j x_j - p_k z_k - z) \frac{\partial f}{\partial p_k} + \Sigma q_j x_j \frac{\partial f}{\partial q_k}, \\
&(\Sigma q_j y_j - z) \Sigma \Big(x_j \frac{\partial f}{\partial x_j} + q_j \frac{\partial f}{\partial q_j} \Big) + (\Sigma x_j p_j - z) \Sigma \Big(y_j \frac{\partial f}{\partial y_j} + p_j \frac{\partial f}{\partial p_j} \Big) \\
&- \Sigma x_j q_j \Sigma y_j \frac{\partial f}{\partial x_j} - \Sigma y_j p_j \Sigma x_j \frac{\partial f}{\partial y_j} + \Sigma y_j p_j \Sigma q_j \frac{\partial f}{\partial p_j} + \Sigma x_j p_j \Sigma p_j \frac{\partial f}{\partial q_j}, \\
&\Sigma x_j \frac{\partial f}{\partial y_j} - \Sigma q_j \frac{\partial f}{\partial p_j}. \qquad \Big(i, k = 1, 2, \ldots, \frac{n-2}{2}, i \neq k. \Big)
\end{aligned}\right\} \quad (12)
$$

We shall denote the $\infty^{\frac{n+1\,.\,n+2}{2}}$ finite transformations of this group by the symbol $G_{\frac{n+1\,.\,n+2}{2}}$. We have then proved the following important

THEOREM: *There exist in* $\overline{\overline{S}}_{n-1}$ $\infty^{\frac{n+1\,.\,n+2}{2}}$ *contact-transformations which transform the* ∞^n *flat spreads*

$$\left.\begin{aligned} x_i &= ay_i + b_i, \\ z &= \Sigma c_i y_i + d, \\ \{q_i &= -ap_i + c_i\}, \end{aligned}\right\} \quad (13)$$

into the ∞^n *spheres in* S_{n-1}. *These transformations are obtained by superposing the inverse of the generalized Lie transformation* L *on the* $\infty^{\frac{n+1\,.\,n+2}{2}}$ *transformations defined by the group of infinitesimal contact-transformations* (12). *The inverse of* L *is*

$$\left.\begin{aligned} x_i &= -(X_{2i-1} + iX_{2i}) - \frac{P_{2i-1} + iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}} X_{n-1}, \\ y_i &= -\frac{P_{2i-1} + iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}}, \quad p_i = \frac{P_{2i-1} - iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}}, \\ q_i &= -(X_{2i-1} - iX_{2i}) - \frac{P_{2i-1} - iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}} X_{n-1}, \\ z &= \frac{\Sigma (P_{2j-1} + iP_{2j})(X_{2j-1} - iX_{2j})}{1 \pm \sqrt{1 + \Sigma P_j^2}} - X_{n-1}.^{*} \end{aligned}\right\} \quad (14)$$

For $n = 4$ *the group* $G_{\frac{n+1\,.\,n+2}{2}}$ *becomes the general projective group in 3-space and* L *the classical transformation of Lie.*

The same transformations $L^{-1}\, G_{\frac{n+1\,.\,n+2}{2}}$ *transform the asymptotic curves on an* $(n-2)$*-spread* M_{n-2} *in* $\overline{\overline{S}}_{n-1}$ *into lines of curvature on the transform of* M_{n-2} *in the space* S_{n-1}.

The differential equations of the asymptotic curves on M_{n-2} *are*

$$\left.\begin{aligned} dx_i dp_{\frac{n-2}{2}} + dy_i dq_{\frac{n-2}{2}} &= 0, \quad i = 1, 2, \ldots, \frac{n-2}{2}, \\ dq_k dp_{\frac{n-2}{2}} - dp_k dq_{\frac{n-2}{2}} &= 0, \quad k = 1, 2, \ldots, \frac{n-4}{2}, \end{aligned}\right\} \quad (15)$$

while those of the lines of curvature on the transform of M_{n-2} *are*

$$(dX_i + P_i dX_{n-1})\, dP_{n-2} - (dX_{n-2} + P_{n-2} dX_{n-1})\, dP_i = 0, \qquad (16)$$
$$i = 1, 2, \ldots, n-3.$$

The last statement may be proved as follows:

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[*] These formulæ are generalizations of those obtained by Darboux in his *Leçons*, Vol. IV, p. 175, for $n=4$. We note that the correspondence is *one-to-two* on account of the double sign of the radical; *i. e.*, to a flat corresponds *one* sphere, while to a sphere correspond *two* flats.

On the $(n-2)$-spread in S_{n-1} the surface-elements along a line of curvature must be functions of a single parameter. If two consecutive surface-elements $(X_i + dX_i,\ P_i + dP_i)$ and $(X_i,\ P_i)$ shall belong to the same curvature-sphere, it is necessary and sufficient that the corresponding consecutive normals at the two points intersect. We have then

$$\bar{X}_i = X_i + tP_i, \quad \bar{X}_{n-1} = X_{n-1} - t, \quad (i = 1, 2, \ldots, n-2),$$

where $\bar{X}_i$, $\bar{X}_{n-1}$ are the running coordinates of the normal. But two consecutive normals will intersect if

$$\begin{aligned} dX_i + P_i\,dt + t\,dP_i &= 0, \\ dX_{n-1} &= dt; \end{aligned}$$

eliminating t and dt from these equations, we obtain the system (16) written above. Q. E. D.

If we introduce Euler's transformation E,

$$x_i = F'_{\alpha'_i}, \quad y_i = \beta_i, \quad p_i = -\alpha_i, \quad q_i = F'_{\beta_i}, \quad z = F - \Sigma\,\alpha_i F'_{\alpha_i},$$

and superpose it on the group $G_{\frac{n+1\,.\,n+2}{2}}$, we obtain $\infty^{\frac{n+1\,.\,n+2}{2}}$ contact-transformations $E\,G_{\frac{n+1\,.\,n+2}{2}}$ which transform the asymptotic lines on M_{n-2} into the Euler lines on the transform in $\bar{S}_{n-1}$. Further, if we introduce the variables α_i, β_i, F'_{α_i}, F'_{β_i} in the group (12), we obtain a group which has the property of leaving invariant the ∞^n quadrics

$$F = a\,\Sigma\,\alpha_j\beta_j + \Sigma\,b_j\alpha_j + \Sigma\,c_j\beta_j + d;$$

it is a subgroup of the group (56) in Vol. II, p. 512, of Lie's *Theorie der Transformationsgruppen.*

If we inspect more closely the group (12), we find that it contains a subgroup of linear transformations, which we write as follows, omitting the terms due to the extension of the transformation:

$$\frac{\partial f}{\partial z},\ \frac{\partial f}{\partial x_i},\ \frac{\partial f}{\partial y_i},\ x_i\frac{\partial f}{\partial z},\ y_i\frac{\partial f}{\partial z},\ x_k\frac{\partial f}{\partial x_k} + y_k\frac{\partial f}{\partial y_k},\ x_k\frac{\partial f}{\partial x_i} + y_k\frac{\partial f}{\partial y_i},$$

$$\Sigma\,y_i\frac{\partial f}{\partial y_i},\ \ \Sigma\,y_i\frac{\partial f}{\partial x_i},\ \ \Sigma\,x_i\frac{\partial f}{\partial y_i},\ \ 2z\frac{\partial f}{\partial z} + \Sigma\left(x_i\frac{\partial f}{\partial x_i} + y_i\frac{\partial f}{\partial y_i}\right).$$

The remaining point-transformations are not projective except in the case $n=4$. Hence, *if* $n>4$ *there exists a group of* $\infty^{\left(\frac{n+2}{2}\right)^2+1}$ *linear transformations which will leave invariant the* ∞^n *flat spreads* (13), *and which transform the* *asymptotic lines on an M_{n-2} into asymptotic lines on its transform.* In these transformations are contained the following $\frac{(n-2)\,(n-4)}{8}$ rotations:

$$x_k\frac{\partial f}{\partial x_i}-x_i\frac{\partial f}{\partial x_k}+y_k\frac{\partial f}{\partial y_i}-y_i\frac{\partial f}{\partial y_k},$$

$$k=1,2,\ldots,\frac{n-2}{2},\quad i=1,2,\ldots,\frac{u-2}{2},\ k\neq i,$$

and the $n-1$ translations $\frac{\partial f}{\partial z}, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_i}$.

Application of the above theory to differential equations will be made in another paper.

WEST VIRGINIA UNIVERSITY, *February* 22, 1912.